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SOLUTION OF A MULTI-ECHELON INVENTORY MODEL WITH POSSIBLE ITEM REPAIR

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United States Naval Postgraduate School

THESIS

SOLUTION OF A MULTI-ECHELON INVENTORY MODEL WITH POSSIBLE ITEM REPAIR

by

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Solution of a Multi-Echelon Inventory Model

with Possible Item Repair

by

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ABSTRACT

This thesis describes briefly the scope and nature of multi-echelon inventory systems where item repair is possible at the various levels. A model is created which describes many commonly encountered real world systems. Sets of non-comparable and comparable costs are specified and a cost-effectiveness approach to the solution of the model is outlined. A nonlinear program is developed where the expected sum of comparable costs is minimized subject to a given level of fill rate (effectiveness) provided by the lower level stocking points to their customers. Several remarks are then made to indicate possible solution procedures for the program and characteristics and uses for the cost-effectiveness alternatives developed.

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I. INTRODUCTION

This thesis deals with a complex model of multi-echelon inventory control in which the items are considered to be repairable. Whenever an item fails, the user demands a replacement and turns in a nonfunctioning unit called a carcass. An attempt is made to repair the carcass and return it to stock for future issue. Characteristics of items designated as repairable are: 1) such items are physically capable of repair, 2) the cost of repair is less than procurement cost, and 3) initial procurement costs are large. Complicating this scheme is the fact that nearly all such models are of the multi-echelon type. That is, several levels of repair and stockage interact in complicated ways which greatly increase the difficulty of analysis. Multi-echelon systems are characterized by a group of stocking activities which receive customer demands and in turn demand resupply from a higher level. There may be any number of levels or echelons; however, in practice only two or three levels are usually encountered. As an example of a multi-echelon inventory system consider a large retail store chain which procures and stores certain types of goods nationally and, when required, supplies regional warehouses which in turn supply local retail outlets. The objectives of the study of multi-echelon inventory control deal to a large extent with the development of those policies of

ordering at all levels which produce some specified system performance, such as minimum expected system operating costs.

In addition to being complex multi-echelon repairable item inventory systems also often entail huge expenditures of money and other resources. One of the largest multi-echelon inventory systems dealing with repairable items is managed by the U. S. Navy's Aviation Supply Office (ASO). Repairables represent 18.7% of the items controlled by ASO by type, for a total of 66,000 separate items; but, almost 60% by dollar value for an investment of nearly 3.1 billion dollars. The Aviation Supply Office spends 16 million dollars per month for repairs at the lower levels of repair alone, that is, simple or relatively inexpensive repairs, and 28 million dollars monthly for procurement of such items. Among ASO managed repairable items, 365,000 units were reworked in fiscal year 1971. In view of these large investments in multi-echelon repairable item systems it is not difficult to understand how a small improvement in the way in which these systems are managed can result in substantial savings.

This thesis describes a model which approximates the actual working structures of many repairable item, multi-echelon inventory systems. A set of costs and measures of effectiveness is defined and its application to the model is discussed. The model is solved for minimum expected total system cost ordering policy subject to a speci ied level of

effectiveness or performance. The solution is presented in the form of a nonlinear programming problem. Several remarks are then made in an attempt to outline possible solution procedures for the program and applications of the model.

II. THE MODEL

The lower level of the multi-echelon repairable item inventory system considered in this thesis will consist of two bases which experience demands resulting from failures of items in use. When a demand is made on base i, a carcass or not-ready-for-issue item is turned in at that base's repair facility. In practice it is the case that some of the carcasses which are turned in at each base cannot be repaired at the base. It will be assumed in this thesis that an item will be repaired at base i with probability p_i. If repair at base i is possible the repaired item is returned to stock at base i after some time, $\underline{rt}_i$, later. It is assumed that $\underline{rt}_i$ is a random variable with some general distribution and has mean rt_i. The repair policy at base i may then best be described as an infinite server queue. If an item is deemed to be unrepairable at a base it is forwarded to a higher level repair facility, depot repair. Define the quantity of items on hand in stock and in repair plus items on order less items ordered by customers but not supplied at base i to be the base i inventory position. When the base i inventory position reaches a level r_i, a request is made to the depot for an additional unit which arrives a time t_i later where t_i is a value of a random variable. Since the base i inventory position only decreases whenever a demand is received and

the accompanying carcass is not repairable, an order is placed only when such a demand occurs.

A single depot receives demands and carcasses from the bases when repair is not possible at base level. It is clear that some items will eventually be lost since their condition will have deteriorated beyond repair. For this reason it is assumed that carcasses received from base i may be repaired with probability q_i and if repair is not possible they are discarded. Items accepted for repair are assumed to be returned to depot stock after a constant repair time, rt. Define the quantity at depot both on hand and in repair to be the depot net inventory. Due to the attrition caused by items being discarded, the depot net inventory position will fluctuate. Define the quantity on hand in stock and in repair plus items placed on order by the depot less orders placed by the bases to the depot but remaining unfilled to be the depot inventory position. When the depot inventory position reaches a level r, an order is initiated by the depot for a quantity Q from a source of manufacture or higher level of supply which arrives after a constant lead time t. This policy is similar to the well known (r,Q) model of single stock points which has many practical advantages as well as intuitively appealing logic and simplicity. Among its advantages is the ease with which it may be understood and applied to nearly any inventory requirement. It is necessary to consider the items in depot and base repairs when developing

an inventory ordering policy. In this thesis this is accomplished by including the quantity in repair in the inventory position. In this manner it is possible to insure that the quantity of items in the entire system remains bounded. The movement of items within the model is illustrated in Fig. 1. A listing of parameters and variables used in the description and solution of the model is included in Table I.

A. MAJOR ASSUMPTIONS

Major assumptions about the model required in its solution are:

> Base i experiences demands which may be closely approximated with a Poisson distribution with rate L_i.
>
> The depot repair facility completes repairs on items in a constant specified time not greater than the procurement lead time t (when repair is possible).
>
> The procurement lead time for the depot is a constant value t.
>
> Bases do not experience crossed orders, i.e. orders are received from the depot in the same sequence in which they were placed.
>
> The depot is never required to place an order to a higher level of supply when a previous order has not been received. This assumption does not usually distort the model since for the usual case where ordering costs are high and holding costs relatively low, orders are placed infrequently.
>
> An order quantity of one is appropriate to a minimum expected cost solution for base i. As argued by Feeney and Sherbrooke [1] , where holding costs at the lower level are typically higher than at the higher level because of such factors as restricted storage capacity

and ordering costs are much lower at the lower level than at the higher level since they only entail administrative and handling costs instead of for example, contract negotiation costs; a policy which orders small amounts often and thus maintains low stock levels usually compares very favorably with the minimum expected cost policy. This is especially true where integer quantities only are considered.

B. PARAMETERS AND VARIABLES

A few comments in explanation are appropriate for some of the parameters and variables listed in Table I. I_{is}, I_{ir}, I and I_r are inventory holding costs for a base i stock, base i repair, depot stock and depot repair respectively. These figures represent the costs composed of such things as storage, handling, loss, breakage, etc. for holding each item of inventory for a given length of time. It is assumed that the holding costs are expressed in units of dollars per item-year. Therefore, the instantaneous rate of incurring carrying charges can be expressed as Ix where x is the on hand inventory level. Although difficult to specify in practice and often not varying linearly with stock on hand, as assumed here, it is considered necessary and acceptable to introduce holding costs in this manner.

The order cost for the depot, A, is taken to be total cost associated with placing, following up and receiving an order. Components of A might be the costs of letting a contract and receiving the order at the warehouse.

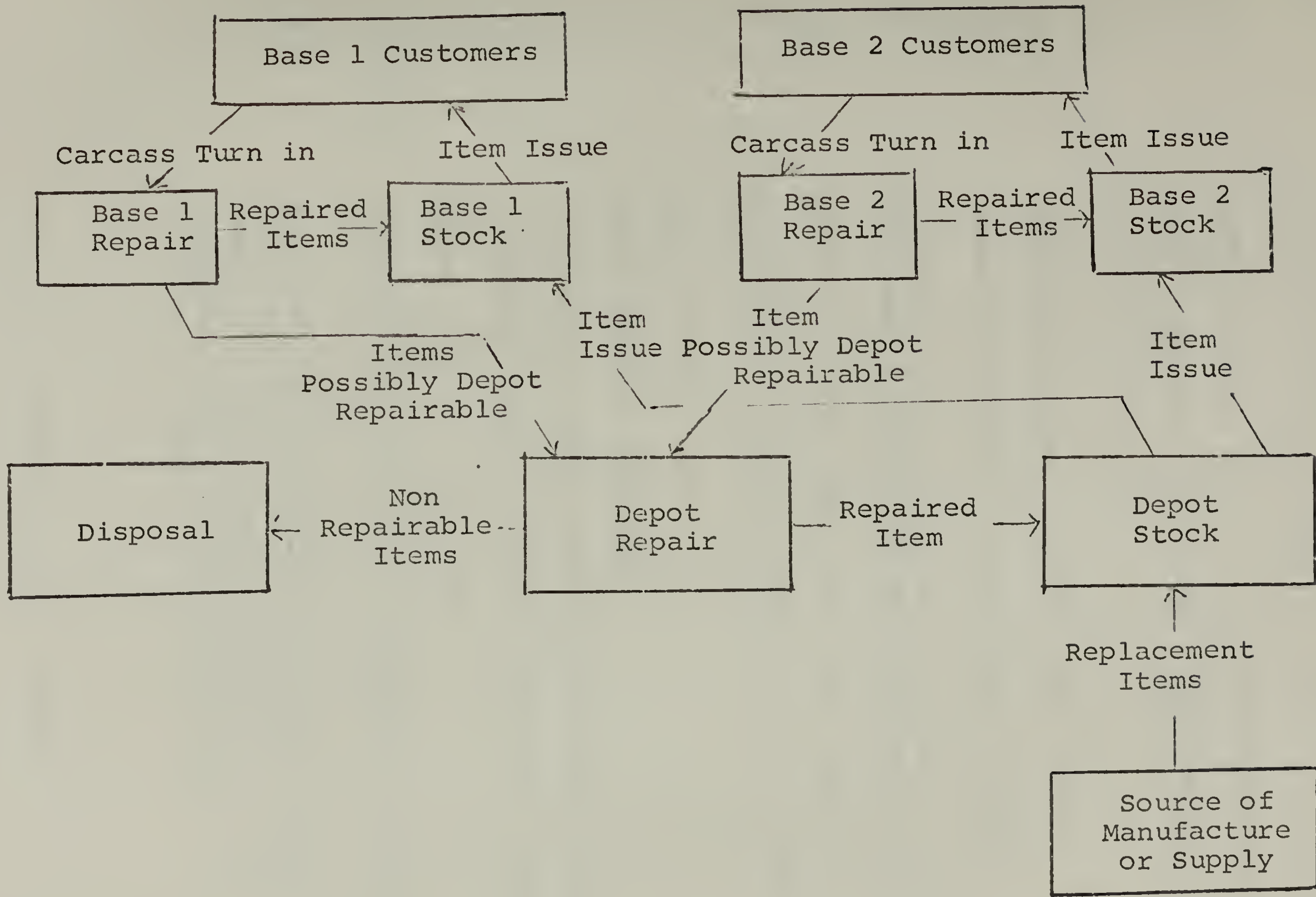

Figure 1. Material flow in a two echelon inventory model with possible repair.

L_i	mean rate of demand for the item experienced at base i
p_i	probability that a carcass turned in at base i may be repaired at base i
rt_i	expected repair time for items repaired at base i
$\bar{t}_i$	mean procurement lead time experienced at base i
r_i	base i reorder level
I_{is}	inventory holding cost for base i stock
I_{ir}	inventory holding cost for base i repair
q_i	probability that an item forwarded to the depot from base i repair can be repaired at the depot
rt	constant repair time for items repaired at depot
t	procurement lead time experienced at depot - a constant
r	depot reorder level
Q	depot reorder quantity
I	inventory holding cost for the depot stock
I_r	inventory holding cost for the depot repair
A	order cost for the depot
$B_i(t_i,r_i)$	probability that base i is able to fill an order without delay whenever a demand is experienced
d_i	expected order/shipping time for base i given that sufficient stock is on hand at depot for an immediate fill - major components of d_i are shipping time and time required for processing of papers, packaging for the order

Table I. Parameters and Variables

III. SOLUTION OF THE MODEL

Solution of an inventory model for an optimal policy is often a very difficult task. First, it is necessary to determine the structure of the problem and what information is available and desired. Usually the analyst proceeds as though the consequences or dollar costs of certain states of the system modeled can be determined; examples are the costs of holding an item in stock without issuing it for a year and the cost of placing an order for some specified number of articles. When all such states can be assigned a cost the investigator then tries to determine the probability of the occurrence of each state of the system and then the expected cost of maintaining the system for a given period of time. It is necessary that this expected cost expression be a function of variables which may be determined by an inventory policy; otherwise, expected operating costs are independent of inventory policy and there is no problem. Most commonly, these policy variables are lead times, order quantities and reorder levels. It is then necessary to solve the expected cost expression for a policy which yields the minimum expected cost. Of course, expected costs may not be of great importance and it might be of value to design a policy which, for example, guarantees with a certain probability that all demands received can be filled within a given period of time. The analogy holds in this case also. The consequences of each state

of the system in terms of the probability of issue within the given period of time must be determined and the probability of filling an order within the given period of time expressed as a function of variables which the inventory policy controls. As in the case with expected costs it is then necessary to solve this problem for a "best" policy which in this case is the one which produces the desired probability that issues are made within the given time period. It is important to realize that the mathematical problem which results from the type of formulation discussed above is unconstrained; that is, any policy which produces the desired value of the single criteria is acceptable.

A problem occasionally arises, however, where it is not possible to assign costs to each possible state which are compatible with costs associated with the characteristics desired. For example, while an investigator may attempt to find policies which minimize total expected system dollar costs over some period of time to operate a system, it is not always possible to assign costs to factors such as not being able to fill a customer demand. What, for instance, is the cost in dollars of not having an item in stock to repair a ship damaged at sea or a meteor struck space craft? In these situations the best that the analyst can usually hope for is to "pass the buck" as far as assigning costs of being out of stock is concerned to the authority responsible for accomplishing the mission involved, i.e. the decision maker. In this formulation the analyst produces for

the decision maker a series of alternatives of expected costs against probability of being able to fill a demand. The alternatives are usually arrived at by solving a group of problems where expected value of the sum of the comparable costs is minimized subject to constraints on the non-comparable costs or consequences; in this case the probability of not being able to fill a demand. It is then up to the decision maker to select a policy which corresponds to his opinions or feelings about the trade off between total system expected cost and payoff, or in the parlance of systems analysis he must specify his preference ordering on the set of measures of cost and effectiveness. If the analyst is able to deduce or is supplied with an expression for the decision maker's trade off policy between cost and effectiveness in this situation, he may, of course, apply this policy to the list of alternatives he has developed and determine a "best" or "optimal" policy. It can be shown [Hadley and Whitin, 1963] that in picking a policy or formulating a preference ordering the decision maker is in effect placing a dollar value cost on not being able to fill a demand and in this sense the two formulations discussed above may be considered equivalent.

In this thesis a formulation similar to the non-comparable costs example discussed above is used. First it is necessary to specify a set of costs to be considered; or stated differently, it is necessary to determine those characteristics of a state of the system which may be

assigned costs which are comparable. Other characteristics which are important but which are not included above are then treated as constraining factors and labeled effectiveness criteria. A set of points in the cost-effectiveness space inferred above is determined by the solution of a program:

> Minimize total expected system costs (comparable) subject to constraints which require that each measure of effectiveness (non-comparable cost) meets or exceeds some stated value

for many different levels of the constraints. Each of these points is simply an alternative presented to the decision maker as in the example noted above. It is then necessary to apply the decision maker's preference order, if available, and solve the related problem for an "optimal" policy.

This relatively elaborate optimization scheme is developed in order to circumvent the very difficult task of specifying the cost of not being able to fill an order immediately, the backorder cost. It is an attempt to place the introduction of this consideration into the system model directly through the decision maker, via his preference ordering. Therefore measures of effectiveness must necessarily address the manner in which backorders are to be treated and more particularly the rate of backorders experienced. A measure of effectiveness is:

> $B_i(\bar{t}_i, r_i)$ - the probability that base i is able to fill a demand immediately from stock on hand. This has been referred to by several authors as the fill rate.

There is one such measure of effectiveness for each base.

The total expected system cost to be minimized is considered to consist of the following comparable costs: total expected order cost for the depot and total expected holding costs summed over bases and the depot. Although these components of total system cost are also difficult to assess in certain situations, they do not present the same possible elements of "operational imperative" as the back-order state as the examples of the damaged ship and space craft illustrate. For this reason and because of the difficulty of arriving at an optimal policy via a preference ordering or decision maker's judgment when the set of measures of effectiveness is large (the decision maker is offered too many alternatives) these costs are assumed specified and included in the objective equation of total system cost. Mathematical expressions for the components of the program are developed below.

A. CONSTRAINTS

Let $B_i(\bar{t}_i, r_i)$ be the probability that base i is able to fill a given demand immediately from stock on hand. Define u_i to be the number of items in repair at base i plus those items on order at base i, v_i to be the number of items in repair at base i repair and w_i to be the number of items on order at base i. Then, since the inventory position of each base is a constant $r_i + 1$ under the one for one ordering policy used by the bases:

$B_i(\bar{t}_i, r_i)$ = P(quantity in base i stock > 0)

$$= \sum_{j=0}^{r_i} p(u_i = j)$$

$$= \sum_{j=0}^{r_i} \sum_{q=0}^{j} p(\, v_i = q,\ w_i = j - q)$$

$$= \sum_{j=0}^{r_i} \sum_{q=0}^{j} \left\{ p(v_i = q)\ p(w_i = j - q) \right\} .$$

This result follows from a finding presented by Richards [3] which shows that the distributions of the number of units in repair and the number of units on order are independent. A summary of his results follows. Item failures or demands placed upon the system in an interval of time $(u, u+s]$ constitute a Poisson process $\left\{ F(s), s \geqq 0 \right\}$. When a failure occurs the item returned to base i is designated base repairable with probability p_i and is, therefore, not base i repairable with probability $1-p_i$. It is assumed that the classification of failed items as base repairable or not base repairable is independent of the process generating failures and is also independent among failed items. In addition, it is easily shown that the number of items designated base i repairable in some interval of time $(u, u+s]$, $F_i(s)$, is Poisson with mean rate $L_i p_i$ and the number of items which are not base repairable in that length

of time is Poisson with mean rate $L_i(1-p_i)$. Also, the distribution of the number of items base i repairable in a length of time given that n total demands have been made in the same length of time is simply binomial with parameters n and p_i. With these facts in mind it is only necessary to show that the number of items base i repairable received in a period of time is independent of the number of items which are not base i repairable received in that same period of time. If this is true, it is well known that the number of items in each classification received in different, possibly overlapping, time intervals is also independent. Consider the following:

$$p\left\{F_2(u+s) - F_2(u) = k,\ F_1(u+s) - F_1(u) = m\right\}$$

$$= p\left\{F_2(u+s) - F_2(u) = k \mid F(u+s) - F(u) = k+m\right\} \cdot p\left\{F(u+s) - F(u) = k+m\right\}$$

$$= \binom{k+m}{k}\left(1 - p_i\right)^k p_i^{\,m} \frac{e^{-L_i s}(L_i s)^{k+m}}{(k+m)!}$$

$$= \frac{\left[(1-p_i)\,L_i s\right]^k (p_i L_i s)^m e^{-p_i L_i s} e^{-(1-p_i)L_i s}}{k!\ m!}$$

$$= \frac{\left[(1-p_i)L_i s\right]^k e^{-(1-p_i)L_i s}}{k!} \quad \frac{(p_i L_i s)^m e^{-p_i L_i s}}{m!}$$

$$= p\left\{F_2(u+s) - F_2(u) = k\right\} p\left\{F_1(u+s) - F_1(u) = m\right\}$$

Because the number of items on order at any time varies only with the receipt of demands accompanied by items which are not base repairable and the procurement lead time, the independence of the distributions of v_i and w_i is established. Because of an important theorem by Palm [4] it is also known that the steady state probability distribution of the number of items in repair is Poisson distributed with parameter $rt_iL_ip_i$ and the number of items on order is Poisson distributed with parameter $\bar{t}_iL_i(1-p_i)$. It is interesting to observe that these distributions depend on the lead times only through their mean values. Thus

$$B_i(\bar{t}_i, r_i) = \sum_{j=0}^{r_i} \sum_{q=0}^{j} \left\{ p(q; rt_iL_ip_i)\; p(j-q; \bar{t}_iL_i(1-p_i)) \right\}.$$

Where

$$p(x;\lambda) = \frac{e^{-\lambda}(\lambda)^x}{x!}$$

B. OBJECTIVE FUNCTION

Let $K(r,Q,r_1,r_2)$ be the total expected system cost for a year. Similarly let $K_i(r_i,\bar{t}_i)$ and $K_d(r,Q)$ be the expected costs for operating base i and the depot respectively. Then $K(r,Q,r_1,r_2) = K_1(r_1,\bar{t}_1) + K_2(r_2,\bar{t}_2) + K_d(r,Q)$.

1. <u>Development of the Depot Expected Cost Expression</u>

Let

$SD(r,Q)$ = E(number of items in stock at the depot)
RD = E(number of items in repair at the depot)
$RSD(r,Q)$ = E(inventory status)
$OD(r,Q)$ = E(number of orders per year)
$BD(r,Q)$ = E(number of backorders at the depot)

where $E(X)$ denotes the expected value of the random variable X, and the inventory status is defined to be the number of items in stock and in repair at the depot less any backordered items.

It can be shown that:

$$K_d(r,Q) = (A)OD(r,Q) + (I_r)RD + (I)SD(r,Q).$$

Since the average annual demand experienced by the depot is for $L_1(1-q_1-p_1) + L_2(1-q_2-p_2) = L$ items the expected number of orders placed by the depot in a year is L/Q where Q is the size of the order placed. It may also be shown using Palm's theorem that the expected number of items in depot repair, RD, is simply $L(rt)$. Since the inventory status includes the number of items on hand plus those in repair less backorders, $SD(r,Q) = RSD(r,Q) - RD + BD(r,Q)$. In addition it can be shown that $BD(r,Q) = LP_{out}$ with probability one where P_{out} is the probability that the number of items on hand is less than or equal to zero. P_{out} is developed in the next section. Therefore, it is only necessary to determine the expected value of the inventory status, $RSD(r,Q)$. This, however, is exactly the expression developed by Hadley and Whitin [2] for the expected on hand inventory for an inventory model which has no repairable feature and experiences demands which are distributed Poisson.

Let Z denote the depot inventory position. It can be shown that Z may assume values $r+1,\ldots,r+Q$. Z does not equal r for any finite length of time since when $Z = r+1$

and a demand is received (recall this may be interpreted as the case when a demand is received without a repairable carcass) an order is placed immediately and $Z = r+Q$. It is also known [Galliher, Morse and Simon, 1959] that the inventory position, Z, is uniformly distributed on the values $r+1, \ldots, r+Q$, i.e. Z takes on values $r+j$ with probability $1/Q$ where $j=1,2,\ldots A$.

Let X be the inventory status. Consider the system at $u - t$ where t is the procurement lead time. Any order placed after $u - t$ will arrive after time u. Any order placed at or before $u - t$ will arrive not later than u. Then; if $Z = r+j$ at time $u - t$ the probability that there are x units on hand at u is the probability that $r + j - x$ items were demanded in time t if $r + j - x \geq 0$ and zero if $r + j - x < 0$. Since demand is Poisson distributed with parameter Lt the probability that $r + j - x$ demands are received in a period t is

$$\frac{e^{-Lt}(Lt)^{r+j-x}}{(r+j-x)!} \qquad \text{if } r + j - x \geq 0$$

$$0 \qquad \text{if } r + j - x < 0.$$

Then

$$p(X = x) = \sum_{j=1}^{Q} \left\{ p(X=x \mid Z = r+j)\; p(Z = r+j) \right\}$$

$$= \frac{1}{Q} \sum_{j=1}^{Q} p(r+j - x;\ Lt) \quad \text{if} \quad -\infty < x < r+1$$

$$= \frac{1}{Q} \sum_{j=x-r}^{Q} p(r+j - x;\ Lt) \quad \text{if} \quad r+1 \leq x \leq r+Q.$$

Therefore:

$$RSD(r,Q) = \sum_{x=-\infty}^{r+Q} p(X = x).$$

It then follows that:

$$SD(r,Q) = \sum_{x=-\infty}^{r+Q} p(X=x) - L(rt) + LP_{out}.$$

2. Development of Base i Expected Cost Expression

By an argument which exactly parallels the one above for the depot it can be shown that:

$$K_i(r_i, \bar{t}_i) = I_{ir}\, p_i\, L_i\, (rt_i) + I_{is} \left[\sum_{x=-\infty}^{r_i+1} p(X_i=x) - p_i\, L_i\, (rt_i) + L_i\, (1 - p_i)\bar{t}_i P_{out_i} \right].$$

This is exactly the expression for $K_d(r,Q=1)$ where the term corresponding to the order cost is omitted since it does not depend on the variables r_i, r and Q, and the demand rate and procurement lead time parameters are those appropriate for Base i.

C. DEVELOPMENT OF THE EXPECTED BASE i PROCUREMENT LEAD TIME, $\bar{t}_i$

It is necessary to determine an expression for $\bar{t}_i$, the expected Base i procurement lead time, which appears in the expressions for $K_i(r_i, \bar{t}_i)$ and $B_i(\bar{t}_i, r_i)$. The procurement lead time for Base i can be decomposed into two parts. The first, called the order and shipping time, is that time which is required to notify the depot of the base's need plus the actual time required to transport the unit from the depot to the base once a unit becomes available. This time will be assumed to have expected value d_i and is independent of the inventory situation at the depot. The second part is the amount of time which the depot requires to obtain a unit to issue. If S denotes the latter time, then it is clear that S will be zero if and only if the on hand inventory at the depot is positive. Thus S is a random variable which depends on the on hand inventory at depot, the items in repair and the items on order at depot.

The expected Base i procurement lead time is, therefore:

$$\bar{t}_i = d_i + E(S).$$

Define the following terms:

W = the random number of items in depot repair

H = the random number of items in depot supply ready for issue less any backorders

$Y = W + H$

R_s = the number of receipts by the depot from procurement and repair in any time interval $(u, u+s]$ of length s

X_s = the number of receipts by the depot from repair in any time interval $(u, u+s]$ of length s

Z_s = the number of receipts by the depot from procurement in any time interval $(u, u+s]$ of length s.

It remains to determine an expression for $E(S)$ as a function of the program variables r and Q to completely specify the program statement sought. For $S \geq 0$, it is true that

$$p(S \leq s) = \sum_{h=0}^{\infty} p(S \leq s \mid H = -h)\, p(H = -h) + p(H > 0)$$

where

$$p(H = -h) = \sum_{x=0}^{\infty} p(Y=x-h)\, p(W = x)$$

and

$$P_{out} = \sum_{h=0}^{\infty} p\left[H = -h\right].$$

As developed in the preceding section it is known that

$$p(Y=k) = \frac{1}{Q} \sum_{j=1}^{Q} p(r+j-k\,;\, Lt), \qquad k \leq r+1$$

$$= \frac{1}{Q} \sum_{j=k-r}^{Q} p(r+j-k;\, Lt), \quad r+1 \leq k \leq r+Q$$

in the steady state. Therefore,

$$p(H = -h) = \sum_{x=0}^{r+h} \left\{ \frac{1}{Q} \sum_{j=1}^{Q} p\left[r+j - (x-h); Lt \right] \cdot p\left[x; (rt) \hat{L} \right] \right\}$$

$$+ \sum_{x=r+h+1}^{r+h+Q} \left\{ \frac{1}{Q} \sum_{j=x-h-r}^{Q} p\left[r+j - (x-h); Lt \right] \cdot p\left[x; (rt) \hat{L} \right] \right\}$$

where $\hat{L} = L_1Q_1 + L_2Q_2$.

Also as shown in the preceding section the random variables X and Y are independent. Therefore, it is possible to write

$$p(H > 0) = \sum_{h=1}^{\infty} p(H=h) = \sum_{h=1}^{\infty} \sum_{x=0}^{\infty} p(Y=h+x)\ p(W=x)$$

$$= \sum_{h=1}^{r} \sum_{x=0}^{r-h} \left\{ \frac{1}{Q} \sum_{j=1}^{Q} p(r+j - (x+h);\ Lt) \cdot p(x;\ (rt)\hat{L}) \right\} + \sum_{h=r+1}^{r+Q} \sum_{x=r+1-h}^{r+Q-h} \left\{ \frac{1}{Q} \sum_{j=x-h-r}^{Q} p(r+j - (x+h); Lt)\ p(x;\ (rt)\hat{L}\ ) \right\}.$$

It remains, therefore, to develop an expression for $P(S \leq s \mid H = -h)$

<u>Case 1</u>: Let $s < t$. (Recall that t denotes the depot procurement lead time.) Whenever $H = -h$, S will be less than or equal to s if and only if R_s is greater than h. Thus,

$$p(S \leq s \mid H = -h) = p(R_s > h) = \sum_{j=h+1}^{\infty} p(R_s=j).$$

Recall that a necessary consequence of the assumption that no more than one order can be unfilled at any time is that only one order can be received in any time interval of length $s < t$. Thus,

$$p(R_s=j) = p(Z_s=0, X_s=j) + p(Z_s=Q, X_s = j-Q).$$

Continuing,

$$p(Z_s = i, X_s = j) = p(Z_s = i)\, p(X_s = j)$$

since the random variables Z_s and X_s are independent.

Proceeding,

$$\begin{aligned} P(Z_s=0) &= P(\text{no order arrives in an interval } (u, u+s] \text{ of length } s) \\ &= P(\text{no items are on order at time } u+s-t) \\ &= \sum_{k=r+1}^{r+Q} p(Y = k) \end{aligned}$$

$$= \sum_{k=r+1}^{r+Q} \left\{ \frac{1}{Q} \sum_{j=k-r}^{Q} p(r+j - k; Lt) \right\} \quad \text{and}$$

$$p(Z_s=Q) = 1 - p(Z_s = 0) \ .$$

Subcase 1: Let $s < rt$. It is true that:

$$p(X_s = j) = \sum_{x=j}^{\infty} p(X_s = j \mid W = x)\, p(W=x).$$

Then

$$P(X_s=j \mid W=x) = P(x-j \text{ items are in repair at time } u+s \text{ each of which was in repair at time } u).$$

An expression for the steady state probability that x items are in repair each of which has been in repair s units of time can be developed for any repair time distribution as follows [Takacs 1962]. Let

$a(s) = P$(any item in repair in the steady state has been in repair at least s units of time).

Then from the distribution of age for a renewal process it is known that

$$a(s) = \left(1/\hat{T}\right) \int_s^{\infty} \left[1 - F(t)\right] dt$$

where $\hat{T}$ is the mean repair time and $F(t)$ is the probability distribution function for the repair time.
Let D_y be the event "y units are in repair at time $u+s$ each of which was in repair at time u" and let X be the number of items in repair at time u. Then

$$P(D_Y) = \sum_{x=j}^{\infty} P(D_Y \mid X=x)\, P(X=x)$$

$$= \sum_{x=y}^{\infty} \binom{x}{y} a(s)^y (1-a(s))^{x-y} \frac{(\hat{L}s)^x e^{-\hat{L}s}}{x!}$$

$$= \frac{a(s)^y (\hat{L}s)^y e^{-Ls}}{y!} \sum_{x=y}^{\infty} \frac{[\hat{L}s\,(1-a(s))]^{x-y}}{(x-y)!}$$

$$= \frac{a(s)^y (\hat{L}s)^y e^{-\hat{L}s}}{y!} e^{\hat{L}s(1-a(s))}$$

$$= \frac{(a(s)\hat{L}s)^y e^{-\hat{L}s\, a(s)}}{y!} = p(y; \hat{L}sa(s))$$

Then,

$$P(X_s = j \mid W = x) = P(x-y; \hat{L}sa(s)).$$

For the case where the repair time is constant, the cumulative distribution function is given by:

$$F(u) = 1 \quad \text{if} \quad u \geq rt$$
$$= 0 \quad \text{if} \quad 0 \leq u < rt.$$

Therefore $a(s) = 1 - s/rt$.

<u>Subcase</u> 2: Let $rt \leq s < t$.

Since $rt \leq s$ everything in repair at time u will have been repaired by time $u+s$. In addition, any items which enter repair in the interval $(u, u+s-rt]$ will also have been repaired. Therefore, the conditional probability that the number of items received from repair in the interval

$(u, u+s]$ is j given that x items are in repair at time u is

$$p(X_s=j \mid W=x) = p(j-x;\ (s-rt)\ L\ ) \quad \text{if } j \geq x .$$

For the case in which $j < x$.

$$p(X_s=j \mid W=x) = 0.$$

Therefore, for $0 \leq s < rt$:

$$p(S \leq s \mid H=-h) = \sum_{j=h+1}^{\infty} \left\{ \sum_{k=r+1}^{r+Q} \left[\frac{1}{Q} \sum_{i=k-r}^{Q} p(r+i-k;\ Lt) \right] \cdot \right.$$

$$\sum_{x=j}^{\infty} p\left[x-j;\ \hat{L}sa(s)\right]\ p\left[x;(rt)\ \hat{L}\right] +$$

$$\left[1 - \sum_{k=r+1}^{r+Q} \frac{1}{Q} \sum_{i=k-r}^{Q} p(r+i-k;\ Lt)\right] \sum_{x=j-Q}^{\infty}$$

$$\left. p\left[X-j+Q;\ Lsa(s)\right]\ p\left[x;\ (rt)\ \hat{L}\right] \right\} .$$

And for $rt \leq s < t$:

$$p(S \leq s \mid H=-h) = \sum_{j=h+1}^{\infty} \left\{ \sum_{k=r+1}^{r+Q} \left[\frac{1}{Q} \sum_{i=k-r}^{Q} p(r+i-k;\ Lt) \right] \cdot \right.$$

$$\sum_{x=0}^{j} p\left[j-x;(S-rt)\ \hat{L}\right]\ p\left[x;(rt)\ \hat{L}\right] +$$

$$\left[1 - \sum_{k=r+1}^{r+Q} \frac{1}{Q} \sum_{i=k-r}^{Q} p(r+i-k;\ Lt)\right] \sum_{x=0}^{j-Q}$$

$$p\left[j-Q-x;\ (s-rt)\ \hat{L}\right]\ p\left[x;\ (rt)\ \hat{L}\right]\Bigg\} .$$

<u>Case</u> 2: Let $s \geq t$.

Because it is assumed that not more than one order is outstanding at any one time it follows that:

$$p\ (S \leq s \mid H=-h) = 1.$$

With the expression for $P(S \leq s)$ thus developed it is possible to calculate:

$$E(s) = \int_0^t p\ (S > s) ds\ .$$

D. STATEMENT OF THE PROGRAM

The minimum expected cost solution for given values of b_1 and b_2 where b_i is the probability that an order occurring at Base i can be filled immediately may be determined by solving the nonlinear program:

$$\text{Minimize} \quad \sum_{i=1}^{2} K_i(r_i,\ \bar{t}_i) + K_d\ (r,Q)$$

subject to

$$B_i(\bar{t}_i,\ r_i) \geq b_i,\ i = 1,\ 2$$

where r_i, r and Q may only assume non-negative integer values.

IV. SOLUTION OF THE PROGRAM

This thesis does not develop an efficient procedure with which the program specified can be solved. However, several observations are offered below in an attempt to delineate the scope and nature of the difficulties involved.

A. PROGRAM TYPE

It should be noted that the program developed is nonlinear in both the objective function and the constraints and the program independent variables are r_1, r_2, r and Q, each of which may assume non-negative, integer values. It is not apparent that the program is convex, and it is almost certain that establishing that the program is or is not convex in the objective equation and/or constraints would involve laborious computations.

B. DEVELOPING A SOLUTION PROCEDURE

It is important to note that for given values of r and Q, r_1 and r_2 are determinate, i.e. they assume the smallest integer values such that $B_1(\bar{t}_1, r_1) \geqq b_1$ and $B_2(\bar{t}_1, r_2) \geqq b_2$, because the objective function which is to be minimized is a non-decreasing function of r_1 and r_2 as are the values of the constraint expression. This indicates that an optimum solution can be arrived at by varying r and Q alone and allowing r_1 and r_2 to assume the smallest integer values such that the constraint expressions are met.

It is reasonable to suppose that due to the nature of the computations involved in the solution of the program, for example, summing various expressions which are easily located in tables, a high speed, digital computer would be employed. While knowledge of convexity can be extremely helpful in locating optimal solutions and in fact assures that any local optimum is global for the problem, the limited nature of the search region and certain special features of the program give grounds for optimism that the program can be solved rapidly.

It should be noted that the bulk of the computations involved in a single iteration of a computer search routine involve determination of the $\bar{t}_i$'s as functions of r and Q. It is probably the case that many of the terms in the expression for $\bar{t}_i$ could be shown to be so small relative to the others that they could be disregarded without loss of essential accuracy. Central to the computation of $\bar{t}_i$ is the expression

$$P(S \leq s) = \sum_{h=0}^{\infty} P(S \leq s \mid H=-h) \quad P(H=-h).$$

While at first glance this appears to involve formidable computations, the physical interpretation of H should not be overlooked. It is highly doubtful that any optimum solution to the program would be characterized by values of r and Q which would produce a significant probability that H be smaller than, for example, -3 or -4, depending on the

program parameters involved. Certainly large probabilities of small values of H would increase $\bar{t}_i$ to a point where the values of r_1 and r_2 required to fulfill the constraints would be prohibitively large, especially when as is usually the case the holding costs at the bases are larger than those at depot. In this example it follows that computation of $P(S \leq s)$ to a degree of accuracy sufficient for the purposes of the program would involve summing over a very few values of the dummy variable h since the remaining terms would be extremely small.

Consider also the regions of r and Q which should be searched. There is reason to suppose that the ranges of values of r and Q which produce minimum expected cost solutions to the program are restricted. Certainly for some value of r (even if $Q=1$) the contribution made by adding an additional item to the reorder point does not significantly decrease the values of the $\bar{t}_i$'s and, therefore, larger values of r need not be considered. A similar argument can be made for Q. There is a value of Q such that the benefit in decreased ordering cost gained by increasing the value of Q does not offset the penalty incurred by increased holding costs which also result. These border values on the regions which are searched for optimum solutions to the program are, of course, extremely sensitive to the program parameter values.

V. APPLICATIONS OF THE MODEL

As noted when developing the model it is often the objective of a constrained optimization problem to develop a group of alternatives from which a decision maker can select a "best" element which reflects his feelings about the trade off between comparable and non-comparable costs. In this thesis an analysis has produced a program which provides a method to obtain points in a space of costs and measures of effectiveness. For a particular inventory item and situation such a set of points might be as shown in Fig. 2 where b_i represents the probability that Base i is able to fill an order on demand, and associated with each set of values of b_1 and b_2 is a cost rounded to the nearest thousand dollars.

b_1							
1.0	9	10	11	12	13	15	17
.9	9	9	10	11	12	14	16
.8	8	9	9	10	12	13	16
.7	6	7	8	10	11	12	14
.6	5	6	8	9	10	11	13
	.3	.4	.5	.6	.7	.8	.9

b_2

Figure 2. A Representative Set of Decision Alternatives

While it is not the goal of this thesis to investigate the various methods which can be used to develop a cost-effectiveness surface, the following points are worthy of note:

The logic of the physical realities of the model dictates that costs will be non-decreasing in b_1 and b_2. If this were not the case it would be possible to "buy" a greater amount of effectiveness for less than is needed to pay for some lesser degree of protection which is contradictory to the minimum cost result of the efficient solution.

In many situations it will only be necessary to develop minimum cost solutions for small ranges of possible values of b_1 and b_2. The reason for this is that in many instances the decision maker can specify in advance, without immediate knowledge of the consequences in terms of effectiveness, the maximum expected cost which can be incurred and/or the general range of effectiveness he wishes to achieve. It may be noted that certain ranges of effectiveness might be of particular interest and that for those areas the effect of a smaller change in the values of b_1 and b_2 could be sought; therefore, it would be necessary to search along a finer grid of values for b_1 and b_2 at least for some particular segment of their ranges. It can be said, therefore, that the ranges of b_1 and b_2 to be searched can be narrowed considerably with knowledge of the decision maker's general policy and the physical realities of the system to which the model is applied.

It might have been noted that terms in the objective equation involving the repair facilities' holding costs were independent of the program variables and, therefore, could have been deleted. These terms were included, however, in order to show the ease with which this model and program can be employed to discover not only efficient inventory ordering policies but also efficient inventory ordering and repair facility management policies. Suppose, for example, it is possible, by incurring an extra cost, to decrease the repair time at the depot by some fraction of the original repair time, that is there exists a function $C(rt)$ defined for rt in some interval $[a,b]$, which is decreasing in rt. It is then possible to specify a new program:

$$
\begin{array}{ll}
\text{Minimize} & K(r_1, r_2, r, Q, rt) + C(rt) \\
\text{subject to} & B_i(\bar{t}_i, r_i) \geq b_i \\
& rt \geq a \\
& -rt \geq -b \\
& i = 1, 2
\end{array}
$$

where r_1, r_2, r and Q are non-negative integers and rt is some real number.

Similar cost terms and constraints can be added for base repairs or any combination of repairs where the repair times can be considered variables. In the case of the bases, however, the additional costs need only be considered to vary with the expected base repair times. It is then possible as above to present the decision maker with a revised

set of costs corresponding to the various degrees of effectiveness achieved.

It may also be noted that while the model configuration developed has two bases, there is no reason why systems with any number of lower echelon units cannot be modeled. The only changes required in the program formulation would be to increase the number of cost terms in the objective function and the number of constraints by a number of terms equal to the additional bases modeled. It may be observed that the model specified can be used to describe a multi-echelon system where item repair is not considered. This is possible by stipulating the probabilities of repair at the various facilities to be zero.

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13. ABSTRACT

This thesis describes briefly the scope and nature of multi-echelon inventory systems where item repair is possible at the various levels. A model is created which describes many commonly encountered real world systems. Sets of non-comparable and comparable costs are specified and a cost-effectiveness approach to the solution of the model is outlined. A nonlinear program is developed where the expected sum of comparable costs is minimized subject to a given level of fill rate (effectiveness) provided by the lower level stocking points to their customers. Several remarks are then made to indicate possible solution procedures for the program and characteristics and uses for the cost-effectiveness alternatives developed.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Inventory						
Repairable Item						
Multi-Echelon						
Inventory Costs						
Measures of Effectiveness						
Comparable and Non-Comparable Costs						
Constrained Optimization						

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